UDC 533.6.011

ON THE DEGENERATION OF SUPERSONIC FLOW DUE TO INTERACTION OF CENTERED COMPRESSION AND RAREFACTION WAVES

PMM, Vol. 43, No. 1, 1979, pp. 45-50

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(Received February 20, 1978)

The isentropic supersonic flow in a duct under conditions of interaction of centered compression and rarefaction waves is considered. Such flow may occur, for instance, in the inlet part of certain optimal asymmetric nozzles [1] and, also, in the case of a deflected supersonic stream. This essentially depends on the behavior of solution of the Darboux equation near the degeneration line for boundary conditions specified at some distance from the latter. It is shown that the considered flow obtains only when the Mach number of the stream in the duct inlet exceeds some value higher than unity. Some numerical results are presented.

1. Let us consider an isentropic plane supersonic flow in a duct defines as follows. A horizontal uniform supersonic stream flows through cross section ao (Fig. 1, a) at Mach number M_1, $M_1 > M_I^* > 1$, where M_1^* is some number which is to be determined. A simple centered rarefaction wave in which $\theta - h(M) = -h(M_I)$, where θ is the angle of inclination of the velocity vector and function $h(M)$ for a polytopic gas with adiabatic exponent $\varkappa$ is of the form

$$h(M) = \lambda \operatorname{arctg}(\lambda^{-1}\sqrt{M^2-1}) - \operatorname{arctg}\sqrt{M^2-1}, \quad \lambda = \sqrt{\frac{\varkappa+1}{\varkappa-1}}$$

Segment op of the lower wall is horizontal and the shape of the wall along pb is such that a simple compression wave cdb is focused at point d at which $\theta + h(M) = \theta_c + h(M_c)$, where $\theta_c = h(M_c) - h(M_I)$. In the region acd the flow parameters are constant: $M = M_c$ and $\theta = \theta_c$. The Mach number along the characteristic db is equal M_I and θ at db is equal $2\theta_c$.

The second duct (Fig. 1, b) differs from the first in that it is curved right from the beginning of the intake section ao. The Mach number M_b at db is, consequently, determined by the following relations [1]:

$$2f(M_1) = f(M_b), \quad f(M) = \left(1 + \frac{\varkappa-1}{2} M^2\right)^{\delta} (M^2-1)^{1/4}$$
$$\delta = -\frac{1}{2(\varkappa-1)}$$

Of interest are the flows in regions pcb (Fig. 1, a) and $qpcb$ (Fig. 1, b) of these two ducts. It can be shown that at fairly high M_I the supersonic flows in these regions of both ducts can be readily determined. On the other hand, it is clear on similarity considerations that at $M_1 = 1$ supersonic flows are unobtainable in these ducts. We can, therefore, presume the existence of numbers $M_1^* = M_I^*(M_c, \varkappa)$, obviously different for each duct, such that it is possible to design supersonic ducts of the type shown in Fig. 1 for $M_I > M_I^*$. Elucidation of this problem will make it possible to indicate the range of admissible numbers M_I for asymmetric nozzles considered in [1].

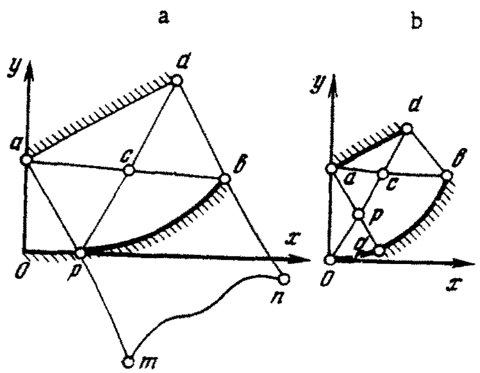

Fig. 1

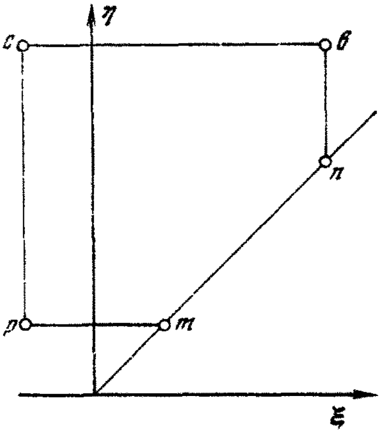

Fig. 2

The flows in regions pcb and $qpcb$ are determined by solving the Goursat problem using the known data along the characteristics pc and cb. We pass to the plane of Riemann invariants. Let $\eta = \theta + h(M)$ and $\xi = \theta - h(M)$ and ψ be the stream function selected so that $\psi_p = 1$ and $\psi_a = 0$. The plane isentropic supersonic flow is defined by the Darboux equation [2, 3]

$$\frac{\partial^2\psi}{\partial\eta\partial\xi} - g(\eta - \xi)\left(\frac{\partial\psi}{\partial\eta} - \frac{\partial\psi}{\partial\xi}\right) = 0 \qquad (1.1)$$

Function $g(\eta - \xi)$ may be represented in parametric form. For a polytropic gas we have

$$\eta - \xi = 2h(M), \quad g(\eta - \xi) = \frac{(\varkappa + 1)M^4}{8(M^2 - 1)^{3/2}} = \frac{1}{6(\eta - \xi)} + q(\eta - \xi) \qquad (1.2)$$

$$|q(\eta - \xi)| < C(\eta - \xi)^{\gamma - 1}, \quad C > 0, \quad \gamma > 0$$

In the plane ξ, η segments pc and cb correspond to characteristics pc and cb (Fig. 2). In the case of the first duct we have

$$\xi_p = -h(M_1), \quad \eta_p = h(M_1), \quad \xi_c = \xi_p, \quad \eta_c = 2h(M_c) - h(M_1)$$
$$\xi_b = 2h(M_c) - 3h(M_1), \quad \eta_b = \eta_c$$

In the case of the second duct point b shifts somewhat to the right without reaching the straight line $\eta - \xi = 0$. Values of ψ on pc and cb (Fig. 2) in terms of η and ξ, respectively, are known and may be represented in parametric form.

The determination of flow in region pcb or $qpcb$ is thus reduced to the Goursat problem for the Darboux equation (1.1) in region $mpcbn$ (Fig. 2). In Fig. 2 segments pm and bn correspond to characteristics pm and bn of the second and first sets in Fig. 1, a. Segment mn of the straight line $\eta = \xi$ in Fig. 2 corresponds to curve mn in Fig. 1, and the Mach number along mn is equal unity.

Three cases are possible.

1°. Along segment mn $\psi > \psi_p = 1$, hence along the line $\psi = \psi_p = 1$ we have $\eta - \xi > 0$, i.e. $M > 1$. In this case a supersonic flow obtains in region pcb ($opcbq$) including the lower wall ob (Fig. 1).

2°. Along segment mn $\psi \geqslant 1$ with the equality satisfied at least at one point. In this case a supersonic flow is still realized in region pcb ($opcbq$) (Fig. 1), but the lower wall contains at least one point at which $M = 1$, i.e. the velocity is sonic.

3°. Along some sections of segment mn $\psi < 1$. In this case no supersonic flow obtains in the considered duct.

It is obviously possible to select M_1 and M_c so as to obtain the first case. The possibility of realizing the second and third cases depends on the behaviour of the solution of the Darboux equation near the degeneration line $\eta - \xi = 0$ (it follows from (1.2) that $g(\eta - \xi) \to \infty$ as $\eta - \xi \to 0$) with the boundary conditions defined away from that line. This problem is dealt with in Sect. 2; we shall only point out here that the solution of Eq. (1.1) with function $g(\eta - \xi)$ of the form (1.2) has a bounded limit when $\eta - \xi \to 0$, provided the boundary conditions on pc and cb are also bounded.

It can be shown that the latter property has the following meaning relative to the considered flows. If for some M_1 and M_c a supersonic flow is realized, i.e. we have case 1°, then, by maintaining M_c constant or varying within some specified limits and reducing M_1 to some $M_1{}^* > 1$, we obtain a duct containing a supersonic flow with sonic points on the lower wall. Further decrease of M_1 leads to Case 3° in which a supersonic flow in the duct is not possible.

Let us illustrate the above on the example of the first duct, assuming that M_1 and M_c are such that a supersonic flow is realized in the duct, but with the supersonic region of influence of characteristics pc and cb is bounded by the straight line $\eta - \xi = 0$ (Fig. 2). Since values of ψ along segment mn are bounded, the truncated square $mpcbn$ in Fig. 2 corresponds in the physical plane to region $mpcbn$ (Fig. 1, a) of finite dimensions, which means that the sonic line mn is at a finite distance from triangle cad a characteristic dimension of the triangle, e.g., the length of segment ad serves as the unit of measurement. For fixed M_c and decreasing M_1 the configuration and position of the sonic line segment remains unchanged in the coordinate system attached to triangle acd, although the sonic line length increases owing to the increase of angle pac. Since the ratio of the length of segment ad to cross section ao tends to zero as $M_1 \to 1$, there exists an $M_1{}^*$ such that when $M_1 = M_1{}^*$ the sonic line reaches the lower wall pb of the duct.

We pass to numerical results obtained by the method of characteristics in the case in which the two ducts represent the inlet parts of asymmetric supersonic nozzles that provide at their outlets a uniform horizontal stream at Mach number M_2 [1]. To obtain this it is necessary to shorten the lower contour at some point s so as to obtain $x_s \leqslant x_b$ and, then, complete the lower and upper contours as shown in Fig. 3. The position of point s must be such as to satisfy the specified condition at point f.

It can be verified that the numbers M_2, M_1, and M_c are linked by the relationship $2h(M_c) = h(M_2) + h(M_1)$.

These calculations were carried out for $M_2 = 4$, $\varkappa = 1.4$, and $x_s \leqslant 2$. Here and

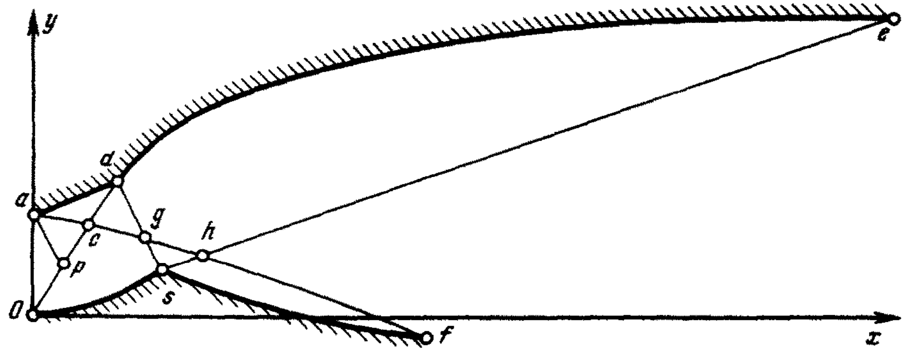

Fig. 3

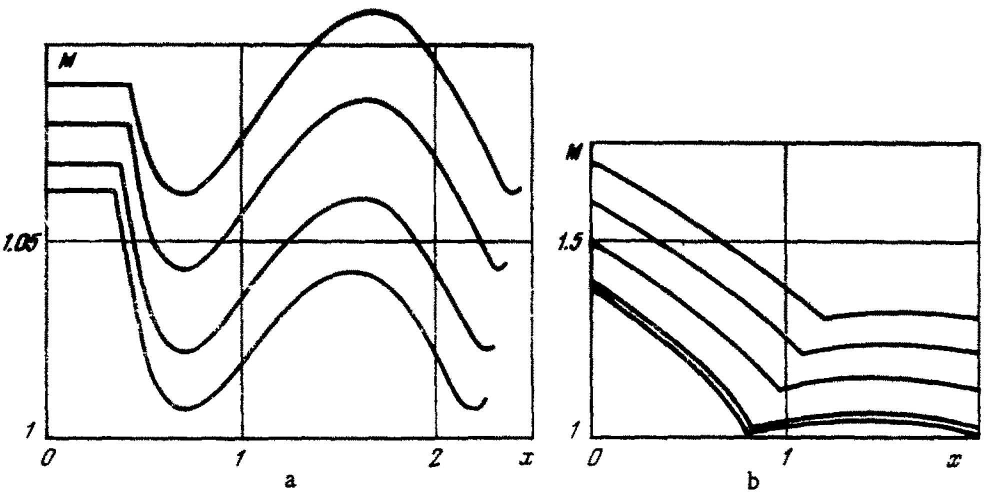

Fig. 4

in what follows linear dimensions are normalized with respect to dimension ao. The dependence on x of the Mach number distribution along os is shown in Fig. 4, where variant a relates to the first duct, and variant b to the second. Note that the corner points on curves relate to points p or q, and that for $x_s > 2$ the curves in Fig. 4, b may also have another minimum. It can be assumed that in the considered cases $M_1^* \approx 1.37$ and $M_1^* \approx 1.06$ for the first and second ducts, respectively. For these values it was at least impossible to carry out calculations even using 600 points on the intake characteristic.

Note that the flow in the region above the streamline passing through point p (Fig. 1, b) is similar to the flow in the first duct. Owing to this, the curves in Fig. 4 and the adduced above Mach numbers M_1^* for the first and second ducts define to a certain extent the rate of "floating up" of the sonic line with decreasing number M_1.

2. Let us investigate the behavior of solution of the Darboux equation near the degeneration line with boundary conditions specified at some distance from that line. We begin by considering the Goursat problem for the Darboux equation

$$\psi_{\eta\xi} - g(\eta - \xi)(\psi_\eta - \psi_\xi) = 0 \tag{2.1}$$
$$\psi = \Phi(\xi), \quad \eta = w, \quad l \leqslant \xi \leqslant k$$
$$\psi = F(\eta), \quad \xi = l, \quad t \leqslant \eta \leqslant w$$

Function $g(z)$ is determinate in the interval $0 < z < w - l$ and satisfies the following conditions: $g(z) > 0$, $0 < z \leqslant w - l$ and $g(z) \to \infty$ as $z \to 0$.

Values of w, t, l, and k satisfy the inequalities $w - l > 0$, $w - k > 0$, $t - l > 0$, and $t - k < 0$. In the above intervals the first derivatives of functions Φ and F are continuous and bounded, hence it is possible to assume that $|F'| \leqslant K$, $|\Phi'| \leqslant K$, and $K < \infty$. Moreover $F(w) = \Phi(l)$.

Below, the "truncated" rectangle $\{l \leqslant \xi \leqslant k, t \leqslant \eta \leqslant w, \eta - \xi > 0\}$ is denoted by W. An example of such region is given by the truncated square in Fig. 2.

We introduce the auxilliary functions f and φ which will be subsequently required

$$f(z) = 2 \int_z^{w-l} g(z)\, dz, \quad \varphi(z) = e^{f(z)}$$

Both functions are determinate for $0 < z \leqslant w - l$ and, if the singularity of function $g(z)$ is integrable with $z = 0$, f and φ are determinate also when $z = 0$.

Theorem 1. Problem (2.1) has a unique solution in region W, except at points $\eta - \xi = 0$. The estimates

$$|\psi_\eta| \leqslant K\varphi(\eta - \xi), \quad |\psi_\xi| \leqslant K\varphi(\eta - \xi) \tag{2.2}$$

are then valid for ψ_η and ψ_ξ. In these estimates K is a constant that bounds from above $|F'|$ and $|\Phi'|$, and $\varphi(\eta - \xi)$ is the function defined above.

Proof. Problem (2.1) can be substituted in region W, except at points $\eta - \xi = 0$ by the equivalent system of integral equations

$$v = \int_l^\xi g(\eta - \xi)(v - u)\, d\xi + F'(\eta) \tag{2.3}$$

$$u = -\int_\eta^w g(\eta - \xi)(v - u)\, d\eta + \Phi'(\xi) \quad (v = \psi_\eta,\ u = \psi_\xi)$$

We solve this system by the method of successive approximations. Let

$$v_0 = F'(\eta), \quad u_0 = \Phi'(\xi) \tag{2.4}$$

$$v_n = \int_l^\xi g(\eta - \xi)(v_{n-1} - u_{n-1})\, d\xi + E'(\eta)$$

$$u_n = -\int_\eta^w g(\eta - \xi)(v_{n-1} - u_{n-1})\, d\eta + \Phi'(\xi), \quad n = 1, 2, \ldots$$

To prove the convergence of these sequences and of their estimates we shall consider the rests $v_{n+1} - v_n$ and $u_{n+1} - u_n$. It can be shown by induction that

$$|v_n - v_{n-1}| \leqslant K \frac{(f(\eta - \xi))^n}{n!}, \quad |u_n - u_{n-1}| \leqslant K \frac{(f(\eta - \xi))^n}{n!} \tag{2.5}$$

Using the definition of function $\varphi(z)$ from (2.5) we obtain

$$\sum_1^\infty |v_n - v_{n-1}| \leqslant K \sum_1^\infty \frac{(f(\eta - \xi))^n}{n!} = K(\varphi(\eta - \xi) - 1)$$

$$\sum_1^\infty |u_n - u_{n-1}| \leqslant K(\varphi(\eta - \xi) - 1)$$

which, in turn, implies that the sequencies v_n and u_n also converge to some limits, and that the estimates (2.2) are valid for $\psi_\eta = v = \lim v_n$ and $\psi_\xi = u = \lim u_n$

Passing to limit in formulas (2.4) we find that the limit functions u and v satisfy system (2.3) and, consequently, are solutions of problem (2.1).

The proof of uniqueness of the derived solution is conventional and is omitted here.

Note that the existence and uniqueness of solution of problem (2.1), in region W, except the band $0 < \eta - \xi \leqslant \varepsilon$ follows from the theorem derived, for instance, in [4]. However the obtained there estimates make it impossible to pass to limit with $\varepsilon \to 0$, since they contain the quantity $A = \max g(\eta - \xi)$ which increases indefinitely with increasing $\eta - \xi \to 0$.

Let us consider some properties of solution when $\eta - \xi \to 0$.

T h e o r e m 2. If function $g(z)$ tends to infinity slower than $z^{-\beta}$, $\beta < 1$ when $z \to 0$, then ψ, ψ_η, and ψ_ξ have finite limits when $\eta - \xi \to 0$. When function $g(z)$ can be represented in the form $g(z) = \alpha z^{-1} + q(z)$, where $|q(z)| < Cz^{\gamma-1}$, $C > 0$, $\gamma > 0$, then for $\alpha < {}^1/_2$ function ψ has a finite limit when $\eta - \xi \to 0$. (We recall that function g of the Darboux equation (1.1) which defines plane supersonic flows of polytropic gas with $\alpha = {}^1/_6$ can be represented in this form).

The validity of both statements follows from estimates (2.2), definition of functions $f(z)$ and $\varphi(z)$, and from that a singularity of the type $z^{-\lambda}$ is integrable as $z \to 0$ and $\lambda < 1$.

The author thanks A. N. Kraiko and V. N. Vragov for their interest in this work and useful discussions.

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Translated by J. J. D.